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CHARACTERIZATION AND RECONSTRUCTION
OF GAUSSIAN PROCESSES
FROM LEVEL CROSSINGS



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av

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THE THESIS CONSISTS OF THIS SUMMARY AND THE FOLLOWING THREE PAPERS:

- [A] de Maré, J.: When are the successive zero-crossing intervals of a Gaussian process independent? Report 1974:3, Dept. of Mathematical Statistics, University of Lund
- [B] de Maré, J.: Reconstruction of a stationary Gaussian process from its sign-changes. Report 1975:11, Dept. of Mathematical Statistics, University of Lund
- [C] de Maré, J.: The behaviour of a non-differentiable stationary Gaussian process after a level crossing. Report 1975:12, Dept. of Mathematical Statistics, University of Lund

The report [A] will be published in IEEE Transactions on Information Theory in a revised form. The research of the reports [B] and [C] has been partly supported by the Swedish Institute of Applied Mathematics and by Umeå University.

1. INTRODUCTION

In a variety of practical problems involving random processes, the processes are only partly observed by technical, economical or theoretical reasons. One example of a practical situation which gives rise to a partly observed random process is the problem of describing the variation of strength along a paper web running through a paper machine. It is then impossible to observe the strength directly and the only information attainable is drawn from web breaks which occur when the strength is less than the web tension level. What is actually observed, are the down-crossings of the tension level of the local web strength (Hållberg & de Maré (1975)).

This dissertation treats some aspects of what can be said about the original process when you just have information on its level crossings. There are two kinds of results which suit this ramification. The first kind is the problem to characterize the possible original processes from their class of level crossing distributions and the second kind is the problem to estimate the process from observations of its level crossings.

A result which belongs to the first kind is given in Paper [A], where it is proved that if level crossings occur as a non-trivial renewal process then the underlying process is not a stationary Gaussian process.

In Paper [B] we are trying to recover a Gaussian process, of which we just know all its sign-changes. This is a problem of the second kind. Also in Paper [C] we treat a problem of the second kind. If we have a stationary and continuous Gaussian process and know a time-point when it crosses a given level, what can then be said of its behaviour conditioned on this knowledge? Slepian (1962) solves this question for differentiable processes and in Paper [C] we discuss the non-differentiable case.

2. ZERO-CROSSING DISTRIBUTIONS

The problem of evaluating the distribution function of the interval between successive zeros is solved only for four essentially different stationary Gaussian processes with mean zero (cf. Blake & Lindsey (1973)). In Paper [A] we treat a reversed problem and find that if the zeros occur as a non-trivial renewal process then the original process cannot be Gaussian. This result contributes to the growing evidence that no simple model can be adequate for the zero-crossing distribution of a non-trivial Gaussian process.

The idea of the proof is that the assumption of the zeros as a renewal process implies that, conditioned on an up-crossing of the zero-level at the origin, the events that the process is positive at time $s < 0$ and that it is positive at time $t > 0$ are independent. It turns out that this implies that these events are independent conditioned on a zero at the origin and on the derivative at the origin as well. By solving a functional equation in the covariance function we show that the only processes which satisfy this condition are the continuous time autoregressive processes of the second order. As McFadden (1958) noted these processes have dependent zero-crossing intervals, which completes the proof.

The clipped process is the process which attains the value one, when the original process is positive and minus one when the original process is negative. It is noted in Paper [A] that although no stationary Gaussian process can give rise to zeros as a non-trivial renewal process there is one clipped Gaussian process which has the same covariance function as the random telegraph signal which just attains the values one and minus one and changes signs at time points given by the Poisson process.

3. RECONSTRUCTION FROM LEVEL CROSSINGS

The problem of giving the best estimation of a stationary and differentiable Gaussian process after a level crossing is solved by Slepian (1962) who gives a complete description of the conditional distribution of the process. He also computed the distribution conditioned on two crossings. In principle it would be possible but tedious to generalize to conditioning on the knowledge of some more crossings than just two.

A related but much deeper question, so far unsolved, is how to condition on the knowledge of crossings at two points and no crossing in between. This question is related to the problem of obtaining the distribution of the interval between two crossings. As is pointed out in Section 2, this problem is just solved in essentially four cases.

Hence it seems difficult to obtain the expectation of the Gaussian process, conditioned on the knowledge of all its sign-changes. This conditional expectation is the best estimator of the process if we just observe the clipped process. In Paper [B] an estimator is proposed which is the best in the class of linear estimators based on the clipped Gaussian process.

Since the clipped Gaussian process itself is a stationary process, stationarily correlated to the Gaussian process, the Kolmogorov-Wiener theory can be used to obtain the best linear estimator. It is shown that this estimator is a convolution of the clipped process and a square integrable deterministic function. To prove this result the problem is transformed to the problem of finding a square integrable solution of a certain integral equation. Elementary Fourier analysis in L^2 is applied to the integral equation together with an approximation based on a local limit theorem by Petrov (1962).

Sufficient conditions are given for the convolution to converge not only in quadratic mean but with probability one. Here the convolution is approximated by a convergent martingale.

The mean square error of the estimator is derived and the behaviour of the reconstruction is illustrated by some simulations.

Slepian's method of obtaining a conditioning on a level crossing is based on the fact that in the regular case every crossing is either an up-crossing or a down-crossing and there are no clusters of crossings (cf. Cramér & Leadbetter (1967) p. 191 and German & Horowitz (1973)). Hence a zero-crossing of the process X at time t occurs precisely when $X(t-\epsilon)$ and $X(t+\epsilon)$ have different signs for all sufficiently small ϵ . The principle difficulty to apply Slepian's method in the non-differentiable case appears from the fact that in this situation every crossing point is a cluster point. In Paper [C] we do the same type of conditioning as Slepian does in the regular case but now the resulting conditioned process has no longer automatically the interpretation of the original process conditioned on a level crossing. However, there is another justification of practical importance if we consider continuous time processes as approximations of discrete time processes. It is shown that the sampled process conditioned on a level crossing converges weakly towards the conditioned process when the sample distance decreases to zero. The space in which the weak convergence appears is that of continuous functions with the topology of uniform convergence on compacts (Whitt (1970)). The discrete time process is considered as a continuous time process by means of linear interpolation between sample points.

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